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THE ONSET OF STOCHASTIC PULSATIONS AT THE EARLY NON-LINEAR STAGE OF THE DEVELOPMENT OF PERTURBATIONS IN THE JET OF AN INCOMPRESSIBLE FLUID PROPAGATING ALONG A WALL†

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The Korteweg–De Vries equation, which describes the non-linear propagation of perturbations in a jet of incompressible fluid emanating from a slit in a planar screen and propagating along a wall is considered. When account is taken of the natural vibrations of the wall, the equation becomes inhomogeneous. If an external action is specified in the form of a running wave, the particular solution of the inhomogeneous equation may be sought in an analogous form. As a result, the simplest problem in the theory of dynamical systems in the Hamiltonian formulation arises. As usual, the existence of a homoclinic structure in the neighbourhood of the separatrices is deduced from an analysis of a Poincaré transformation. Among the trajectories belonging to the homoclinic structure in the secant plane, there are some with properties which are formulated in terms of determinate chaos. A fundamentally important conclusion concerning the dual role of solitons at the non-linear stage of the wave motion of the fluid follows: on the one hand, they serve as the nuclei of large-scale coherent structures and, on the other hand, they are responsible for the onset of stochastic pulsations.

LET A planar jet of viscous incompressible fluid emerge from an aperture in a screen into a space where a fluid with the same physical properties is at rest. We will assume that there is a rigid wall, which bounds the jet from below and is arranged perpendicular to the screen. For simplicity, we will assume the wall to be a plate and take the distance from the leading edge up to the point being considered on its surface as the characteristic length. We will assume that the plate coincides with the y' axis of a Cartesian system of dimensionless coordinates x', y'. By virtue of the no-slip condition on the surface of the plate, the two components of the velocity vector u' and v', referred to the maximum velocity of the fluid particles in the jet, vanish. As the outer edge of the jet, $y'\to\infty$, is approached, the velocity also tends to zero. The bulk of the jet constitutes an unusual boundary layer.

Similar motions of a viscous fluid occur on a rotating disc [1] and, also, near a heated vertical plate [2]. In the latter case, the fluid floats upwards under the action of the buoyancy force. The special features which arise in the velocity field in the neighbourhood of the edges of rigid surfaces were pointed out in [3] and the steady separation of the jet and the subsequent development of a zone of recirculating flow have been studied in [4]. The stability of the fluid motions, which had been constructed in [1, 2], with respect to long-wave perturbations was analysed in [5].

Results which refer to the fine structure of the velocity field have been obtained in [3–5] within the framework of modern boundary-layer theory under conditions of free interaction with an inviscid flow. This theory is based on the expansion of the required functions in asymptotic series in powers of a small parameter $\epsilon = R^{-1/14}$, where the Reynolds number $R\to\infty$. In the viscous sublayer adjacent to the wall past which the flow occurs, the excess pressure p', which is measured from the pressure in the space surrounding the jet and reduced with respect to twice the velocity head, is of the order of ϵ^4 while the longitudinal component u' and the transverse component v' of the dimensionless velocity vector are estimated by means of ϵ^2 and ϵ^5,

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respectively. As a result, the problem reduces to the integration of the Prandtl equations in which the gradient of the self-induced pressure has to be determined jointly with the velocity field. The relationship between the excess pressure and the displacement thickness $-A'$, which also appears in the boundary condition on the outer edge $y' \to \infty$ of the boundary sublayer, closes the problem.

The situation fundamentally changes when the order of magnitude of the dimensionless excess pressure becomes greater than ϵ^4, which causes a corresponding increase in the values of the longitudinal and transverse velocity components of the perturbed flow. An asymptotic theory of waves with increased amplitude, which rests on the earlier assumption that $R \to \infty$, has been developed in [6] and was subsequently considered in [7, 8]. Its basic consequence reduces to the fact that, in its turn, the boundary domain of a jet is stratified into two sublayers with quite different properties. The role of viscous tangential stresses is negligibly small in the upper of the two sublayers which are formed, and the term $\partial^2 u'/\partial^2 y'$ drops out of the Prandtl equations here. The system of equations which has been simplified in this manner admits of a simple integral which satisfies all the boundary conditions of the problem if the displacement thickness is defined by the Korteweg–De Vries (KDV) equation. The latter equation is homogeneous in the case of a wall with an initial position $y' = 0$. However, when account is taken of the natural oscillations of the wall in the form $y' = y'_w = h'(t', x')$, it turns into an inhomogeneous KDV equation.

By means of a normalized system of units, where the former notation with the omitted prime as a superscript is used for the independent variables and the unknown functions, we get [6–8]

$$\frac{\partial A_h}{\partial t} + A_h \frac{\partial A_h}{\partial x} = \frac{\partial^3 A_h}{\partial x^3} - f_0(t, x)$$

$$p = -\frac{\partial^2 A}{\partial x^2}, \quad A_h = h + A, \quad f_0 = \frac{\partial^3 h}{\partial x^3} \tag{1}$$

It is convenient to take the outer perturbation in the form $f_0 = B_0 f(\nu t, \alpha x)$, which shows the amplitude B_0 as well as the frequency and wave parameters ν and α explicitly. We now carry out the change of independent variables $t \to \nu^{-1} t$, $x \to \alpha^{-1} x$ and introduce the new required function $A_h \to \alpha^2 A_h$. From (1), we then obtain the equation

$$\omega \frac{\partial A_h}{\partial t} + A_h \frac{\partial A_h}{\partial x} = \frac{\partial^3 A_h}{\partial x^3} - Bf(t, x) \tag{2}$$

in which only two positive constants: $\omega = \nu/\alpha^3$ and $B = B_0/\alpha^5$ appear.

Let us now consider a wall under the action of a wave propagating along it in the direction of the jet flow. In this simplest case, it is possible to assume that $f = f(\xi)$, $\xi = x - t$ and also to seek a solution of Eq. (2) in the form of a running wave $A_h = F(\xi)$. As the result of a single integration of the ordinary differential equation which defines F, we find

$$d^2F/d\xi^2 + \omega F - \tfrac{1}{2}F^2 = C + BI(\xi), \quad I = \int f(\xi)\, d\xi \tag{3}$$

where C is an arbitrary constant. This is an equation for the forced vibrations of a non-linear oscillator with a quadratic restoring force and, since there is no term with $dF/d\xi$, there is no dissipation of energy during the operation of the oscillator. Let us write (3) in the form of a system of canonical equations

$$\begin{aligned} dF/d\xi &= \partial H/\partial P = P \\ dP/d\xi &= -\partial H/\partial F = \tfrac{1}{2}F^2 - \omega F + C + BI(\xi) \end{aligned} \tag{4}$$

in which the Hamilton function

$$H = H_0(F, P) + BH_1(F, \xi) = {}^1/_2 P^2 - {}^1/_6 F^3 + {}^1/_2\, \omega F^2 - CF + BFI(\xi)$$

is used.

It is now possible to make use of well-known results of the modern theory of oscillations, having first turned to the limiting case when $B \to 0$. For simplicity, let us assume that the wave, which runs along the wall with the fluid jet adjacent to it, is harmonic and that the function $I = \sin\xi$ corresponds to it.

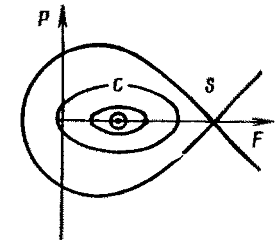

FIG. 1.

Let us start with a phase picture of system (4) when $B = 0$. In this case, the Hamilton function is explicitly independent of ξ, and system (4) therefore admits of a first integral $H_0(F, P) = \text{const}$ and, as can be seen from Fig. 1, is characterized by two equilibrium positions: $F = F_0(\xi, 0)$, $P = P_0(\xi, 0)$ and $F = F_\pi(\xi, 0)$, $P = P_\pi(\xi, 0)$ which are described by the singular points

$$F_0 = \omega - \sqrt{\omega^2 - 2C},\ P_0 = 0 \tag{5}$$

$$F_\pi = \omega + \sqrt{\omega^2 - 2C},\ P_\pi = 0 \tag{6}$$

Their existence is due to the inequality $\omega^2 > 2C$ which is assumed to be satisfied. The closed trajectories which surround the centre C are periodic solutions of the initial KDV equation and the loops with the separatrices of the saddle S are solitons which propagate along the jet with respect to the positive background $F_\pi = \omega + \sqrt{\omega^2 - 2C}$, $P_\pi = 0$ for any value of the constant C. The equation of the separatrices in parametric form is obtained from the well-known relationship which specifies the form of a soliton $F = F_{sol}(\xi, 0)$, $P = P_{sol}(\xi, 0)$. According to [10], for example, we have

$$\begin{aligned} &F_{sol} = F_\pi - 3\sqrt{\omega^2 - 2C}\,\mathrm{ch}^{-2}\eta \\ &P_{sol} = 3(\omega^2 - 2C)^{3/4}\,\mathrm{sh}\,\eta\,\mathrm{ch}^{-3}\eta,\ \eta = (\omega^2 - 2C)^{1/4}\xi/2 \end{aligned} \tag{7}$$

It is of fundamental interest to answer the question as to which motions can arise in system (4) and, in particular, which involve the separatrices of the saddle point when $B \neq 0$.

Results referring to the neighbourhood of the centre C are formulated most simply. According to Poincaré's theorem [11], a family of periodic solutions $F = F_0(\xi, B)$, $P = P_0(\xi, B)$ of the system of Eqs (4) exists for sufficiently small values of the parameter B, and this family is analytically dependent on this parameter and reduces, when $B = 0$, to solution (5) subject to the condition that

$$(\omega^2 - 2C)^{1/4} \neq j,\ j = 0, \pm 1, \ldots \tag{8}$$

In the case under consideration, the processes occurring in the system are actually weakly non-linear and, as regards the initial fluid jet propagating along a wall, they are realized in the small amplitude waves which run along it. Condition (8) serves to preclude resonances.

Similar results also hold for the neighbourhood of the saddle point S. On the basis of Poincaré's theorem which has been cited above, it may be asserted that the family of periodic solutions $F = F_\pi(\xi, B)$, $P = P_\pi(\xi, B)$, which arises for sufficiently small values of the parameter B, is analytically dependent on this parameter and reduces to solution (6) when $B = 0$. Since the non-linear term $\frac{1}{2}F^2$ on the right-hand side of the second of Eqs (4) turns out to have a decisive effect on the course of the whole process, there are no additional conditions of the type (8) which preclude the existence of resonances. The above-mentioned periodic solutions describe pulsations which develop in the unstable zone (6). In the case of the initial KDV equation, this background is the limiting state of jets which is attained by them as a result of a growth in the displacement thickness.

Let us now investigate what takes place with the separatrices of the saddle point when the parameter B is non-zero but remains quite small. As was noted above, when $B = 0$, the loops of integral curves with the separatrices of the saddle serve to depict the solitons of the KDV equation. The introduction of an external periodic force into system (4) therefore enables one to indicate states into which a soliton can pass under the action of a wave which is running along the wall bounding the jet.

Up to now, explicit use has not been made of the fact that one of the separatrices is stable in the neighbourhood of the saddle S while the other is unstable (locally, we mean by stable a separatrix occurring in S as $\xi \to \infty$ and, by unstable, a separatrix originating from S as $\xi \to -\infty$). Let us denote the first by Λ_s and the second by means of Λ_u and let us distinguish two branches in each of them: Λ_s^+, Λ_s^- and Λ_u^+, Λ_u^- in accordance with the position of these branches in the upper or lower half-planes of the phase plane F, P (see Fig. 1). The continuations of the branches Λ_s^+ and Λ_u^- which form the loop yield one and the same solution (7), that is, they are actually coincident. The saddle point itself can be treated as an unstable equilibrium state of hyperbolic type. For sufficiently small value of B, the periodic solutions $F = F_\pi(\xi, B)$, $P = P_\pi(\xi, B)$ will then also be hyperbolic. In the three-dimensional phase space ξ, F, P, the separatrices become invariant surfaces $W_s(B)$ and $W_u(B)$. Trajectories (integral curves) along the first of them tend towards a periodic motion while those along the second depart from it as $\xi \to \infty$. When $B = 0$, the two surfaces $W_s(B)$ and $W_u(B)$ merge and transform into a single cylindrical surface, the shape of which can be obtained by passing a generatrix parallel to the ξ-axis through each point of the loop of separatrices consisting of the Λ_s^+ and Λ_u^- branches together with their extensions. The mutual position of the surfaces $W_s(B)$ and $W_u(B)$ with $B \neq 0$ determines the possible classes of solutions $F = F_{\text{sol}}(\xi, B)$, $P = P_{\text{sol}}(\xi, B)$.

A measure of the distance between the stable and unstable manifolds being considered is given by the Mel'nikov function [12]

$$J(\varphi_0) = \int_{-\infty}^{\infty} \left\{ H_0[F_{\text{sol}}(\xi), P_{\text{sol}}(\xi)],\ H_1[\xi + \frac{\varphi_0}{\omega},\ F_{\text{sol}}(\xi)] \right\} d\xi$$

where the braces, as usual, denote canonical Poisson brackets. As applied to the system of equations (4), we find by direct calculations

$$J(\varphi_0) = 3\sqrt{\omega^2 - 2C}\cos\varphi_0 \int_{-\infty}^{\infty} \frac{\text{sh}\,\eta}{\text{ch}^3\eta} \sin\frac{2\eta}{(\omega^2 - 2C)^{1/4}} d\eta =$$

$$= 12\pi \frac{\text{sh}\,\pi(\omega^2 - 2C)^{-1/4}}{\text{ch}\,2\pi(\omega^2 - 2C)^{-1/4} - 1} \cos\varphi_0$$

The equality $\cos\varphi_0 = 0$ fixes the simple isolated zeros $\varphi_0 = (n + \frac{1}{2})\pi$ ($n = \ldots -1, 0, 1, \ldots$) of the Mel'nikov function. It follows that, when $B \neq 0$, the stable surface $W_s(B)$ and the unstable surface $W_u(B)$ are split and intersect, and that the system of equations (4) does not have a first integral which is analytically dependent on ξ and the phase variables F and P.

It is useful to formulate the results which have been obtained in terms of a Poincaré transform by finding the dependence of the coordinates of the points (F, P) in the secant plane $\xi = 2\pi(n + 1)$ as a function of the coordinates (F, P) in the $\xi = 2\pi n$ plane. All the planes $\xi = \text{const}$ which are formed in this manner are displaced by an integer number of periods of the perturbing force and we shall therefore identify them with one another. The possibility of speaking of a point mapping of a plane $\xi = \text{const}$ into itself follows from this. The cutting of the surfaces $W_s(B)$ and $W_u(B)$ by the plane $\xi \bmod 2\pi = \text{const}$ being considered yields curves which are formed as a result of the successive application of the transform. It is customary to refer to the curves as separatrices [13]. Stable and unstable separatrices have a denumerable set of common points. These points of intersection of the separatrices in the secant plane belong to double asymptotic trajectories which were called homoclinic trajectories by Poincaré. Their neighbourhood, for which the term homoclinic structure is used, contains an infinite variety of trajectories and therefore defines the extremely complex dynamics of the system under consideration in spite of its relatively simple form. Methods of studying stochastic processes are typically employed to investigate non-linear oscillation in this region. An analysis of such oscillations in terms of symbolic dynamics has been developed in [14].

We will now point out some fundamental consequences. The simplest bounded solutions of system (4) are periodic integrals with a cycle, the magnitude of which is a multiple of the period of the perturbing force, and their existence follows from what has been said above. Among the bounded solutions, there are also aperiodic solutions and, within those, the phase variables F and P change within fixed limits which depend on the amplitude B of the perturbing force. The doubly asymptotic solutions which have been mentioned belong to this set, and orbits for which, during motion along them, a point with the coordinates $F = F(\xi)$, $P = P(\xi)$ completes a single complete revolution as ξ changes from $-\infty$ to ∞ correspond to two such solutions.

System (4) admits of various types of unbounded integrals. Among those, there are some which increase without limits as $\xi \to -\infty$ but are bounded as $\xi \to \infty$. Moreover, the range of variation of F and P remains finite for non-asymptotic motions in the case of sufficiently large values of ξ and becomes small in the case of asymptotic motions if $B \to 0$. Unlike those, there are integrals which are bounded as $\xi \to -\infty$ but increase without limit as $\xi \to \infty$. In those integrals, the phase variables F and P vary within an interval which remains finite in the case of non-asymptotic motions and becomes small in the case of asymptotic motions in the limit as $B \to 0$. Finally, system (4) possesses solutions which are unbounded as $\xi \to \pm\infty$ in which F and P execute oscillations within finite limits in any sufficiently large range of ξ which has been specified in advance.

Apart from this, a complete analysis of the trajectories belonging to a homoclinic structure in the secont plane $\xi \bmod 2\pi = \text{const}$ leads to a fundamentally important conclusion regarding the development of dynamics which is referred to as determinate chaos. In order to explain the basic properties of the orbits occurring in the exceedingly complex dynamics, let us fix the accuracy in the specification of the initial conditions for system (4). It is then possible to specify a set of such conditions which are identical at the adopted accuracy to one another and define solutions of system (4) which differ not only quantitatively but also qualitatively. It is clear that the behaviour of the orbits corresponding to these solutions will be random in the three-dimensional phase space ξ, F and P.

We will now make use of the established properties of the solutions of a Hamiltonian system in order to describe the wave motions which can arise in a jet of an incompressible fluid if the wall bounding it from below executes harmonic oscillations. The basic conclusion lies in the fact that, as soon as the perturbations enter the substantially non-linear stage of their development, large-scale ordered structures are formed in them which are determined by the solitons of the KDV equation. The break-up of these structures under the action of an oscillating plate is accompanied by the generation of various, more-complex forms of motion and, in particular, random pulsations. The latter arise in the bulk of a laminar flow as a result of the further natural development of non-linear processes which lead to a pronounced distortion of the initial Tollmin–Schlichting waves. Hence, the appearance of narrow surges of comparatively large amplitude in the traces of signals in real oscillograms serves to foreshadow the commencement of the stochastization of the perturbation fields. It is characteristic that random components in the motion of a fluid arise very early on and their subsequent amplification leads to the transition of laminar flow into turbulent flow. On the other hand, the formation of new ordered structures which are extremely close to solitons is possible if the amplitude of the perturbing oscillations tends to zero.

Amplification of broad band noise as a result of its non-linear interaction with an initially monochromatic Tollmin–Schlichting wave is characterized by related features. Experiments with a Blasius boundary layer on a flat plate placed in a stream of incompressible fluid have been carried out recently [15]. In this case, the propagation of perturbations at the substantially non-linear stage, which corresponds to the analysis carried out above, obeys the Benjamin–Ono integro-differential equation [6, 7]. With the generally accepted notation, instead of (1), we have

$$\frac{\partial A_h}{\partial t} + A_h \frac{\partial A_h}{\partial x} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 A_h}{X - x} dX - f_0(t, x) \tag{9}$$

$$A_h = h + A, \quad f_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 h / \partial X^2}{X - x} dX$$

As was done earlier, we will write the outer perturbation by means of $f_0 = B_0 f(\nu t, \alpha x)$ with the amplitude B_0 as well as the frequency and wave parameters ν and α shown explicitly. By making the replacement of the independent variables $t \to \nu^{-1} t$, $x \to \alpha^{-1} x$ and introducing the new required function $A_h \to \alpha A_h$, we obtain the standard equation

$$\omega \frac{\partial A_h}{\partial t} + A_h \frac{\partial A_h}{\partial x} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 A_h/\partial X^2}{X - x} dX - Bf(t, x)$$

from (9). in which two positive constants: $\omega = \nu/\alpha^2$ and $B = B_0/\alpha^3$ occur. Let $f = f(\xi)$ and $\xi = x - t$. We shall then also seek the solution of the latter equation in the form of a travelling wave $A_h = F(\xi)$. As a result of a single integration of the equation defining F we find

$$\frac{1}{\pi} \frac{d}{d\xi} \int_{-\infty}^{\infty} \frac{F(\zeta)}{\zeta - \xi} d\zeta - \omega F + \tfrac{1}{2}F^2 = C - BI(\xi), \quad I = \int f(\xi)\, d\xi \qquad (10)$$

where C is an arbitrary constant. Since

$$\int_{-\infty}^{\infty} e^{\pm i\zeta} \frac{d\zeta}{\zeta - \xi} = \pm\, i\pi e^{i\xi}$$

it may be assumed that the function $I = \sin\xi$.

Equation (10) has been successfully used to explain the nature of the short-scaled large-amplitude surges mentioned above which are generated by a source which is harmonic with respect to time in an incompressible Blasius boundary layer and revealed in the oscillograms of periodic pulsations [16]. In the limit, periodic solutions give solitons as occurred in the case of the KDV equation. Equation (10) can therefore be used to study the possible forms of motion which arise when the algebraic Benjamin–Ono solitons break down. This obviously also yields a mathematical model, the search for which has been discussed [15] in connection with the development of chaotic pulsations under the action of broadband noise on a Tollmin–Schlichting wave at the non-linear stage in the growth of the amplitude. However, the model proposed is significantly more complex than that which is the basis of the analysis which has been carried out, since it does not reduce to a Hamiltonian system of equations on account of the integral term on the left-hand side of (10). The latest experimental data [17] are evidence in favour of the closeness (in a qualitative connection) of the non-linear processes in a jet propagating along a wall and a Blasius boundary layer.

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